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Griffith formulae for elasticity systems with unilateral conditions

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Abstract: In the paper we consider the elasticity equations in nonsmooth domains in $R^n, n = 2, 3$. The domains have a crack whose length may change. At the crack faces, inequality type boundary conditions describing a mutual nonpenetration of the crack faces are prescribed. The derivative of the energy functional with respect to the crack length is obtained. The Griffith formulae are derived in 2D and 3D cases and the other properties of the solutions are established. In two-dimensional case the Rice–Cherepanov's integral over a closed curve is constructed. The path independence of the Rice–Cherepanov's integral is shown.

Key-words: Griffith formula, Rice–Cherepanov's integral, unilateral conditions on crack faces

(Résumé : tsvp)

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La formule de Griffith et l'intégrale de Rice–Cherepanov pour des équations élliptiques avec conditions unilatérales dans des domaines non réguliers

Résumé : La formule de Griffith est obtenue dans le cas de conditions unilatérales sur la fissure. L'indépendance par rapport au chemin de l'intégrale de Rice–Cherepanov est prouvée. L'indépendance par rapport au chemin de l'intégrale de Rice–Cherepanov était précédemment prouvée en élasticité pour les conditions aux limites linéaires $\sigma_{22} = 0, \sigma_{12} = 0$ sur $\Xi_l^\pm$ dans le cas bidimensionnel. La formule de Griffith est aussi obtenue dans le cas tridimensionnel. Dans ce cas, la fissure bidimensionnelle est incluse dans un domaine tridimensionnel.

Mots-clé : formule de Griffith, fissure, intégrale de Rice–Cherepanov, élasticité

1. Two-dimensional case.

Let $D \subset R^2$ be a bounded domain with a smooth boundary Γ, and $\Xi_{l+\delta}$ be the set $\{(x_1, x_2) | \; 0 < x_1 < l + \delta, \; x_2 = 0\}$. We assume that this set belongs to the domain D for all sufficiently small δ, and $l > 0$.

The domains with cracks $\Xi_{l+\delta}$, Ξ_l are denoted by $\Omega_\delta = D \backslash \overline{\Xi}_{l+\delta}$, $\Omega = D \backslash \overline{\Xi}_l$, respectively.

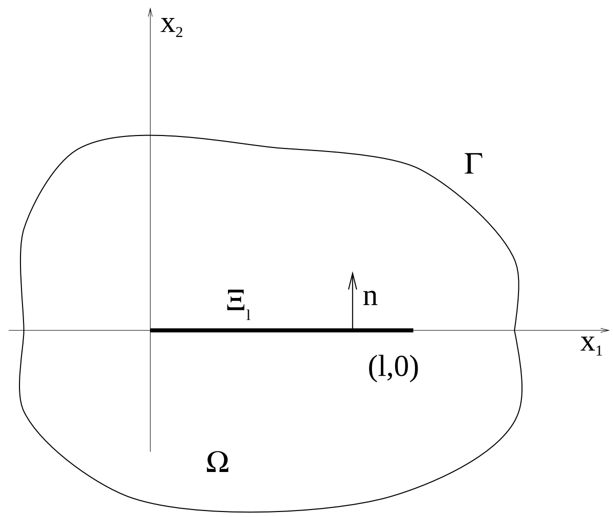

Figure 1: Domain Ω.

The elasticity system analysed in the paper can be formulated as follows. We want to find a function $W = (u, v)$ such that

$$-\sigma_{ij,j} = f_i, \quad i = 1, 2, \quad \text{in } \; \Omega, \tag{1.1}$$

$$W = 0 \quad \text{on } \; \Gamma, \tag{1.2}$$

$$[W]n \geq 0 \quad \text{on } \; \Xi_l. \tag{1.3}$$

Here $\sigma_{ij} = \sigma_{ij}(W)$ are the stress tensor components, the summation convention is used over the repeated indices $j = 1, 2$, $n = (0, 1)$ is a normal vector to Ξ_l, and $[W] = W^+ - W^-$ is the jump of W across Ξ_l. The signs $\pm$ correspond to the positive and negative directions of n. Hooke's law is assumed to be fulfilled,

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda \operatorname{div} W \delta^i_j, \quad i, j = 1, 2. \tag{1.4}$$

By $\lambda \geq 0$, $\mu > 0$ we denote the Lame parameters, $\varepsilon_{ij} = \varepsilon_{ij}(W)$, δ^i_j is the Kronecker symbol,

$$\varepsilon_{11} = u_{x_1}, \quad \varepsilon_{22} = v_{x_2}, \quad \varepsilon_{12} = 1/2\,(u_{x_2} + v_{x_1}). \tag{1.5}$$

We assume that $f = (f_1, f_2) \in C^1(\bar{D})$. The formulation of the problem (1.1) – (1.3) is not complete. Actually, we define a variational solution by the minimisation of the functional

$$I(\Omega; U) = \frac{1}{2}\int\limits_{\Omega} \sigma_{ij}(U)\varepsilon_{ij}(U) - \int\limits_{\Omega} fU, \quad U = (u, v), \tag{1.6}$$

over the set

$$K_0 = \{(u, v) \in H^1(\Omega) \mid \quad u = v = 0 \quad \text{on} \ \ \Gamma, \quad [v] \geq 0 \quad \text{on} \ \ \Xi_l\}. \tag{1.7}$$

In this case the solution W of the minimisation problem satisfies (1.1) – (1.3) and some additional boundary conditions prescribed on Ξ_l. These conditions are analysed below (see (1.28)).

The perturbed problem corresponding to (1.1) – (1.3) is defined as follows. We want to find a function $W^\delta = (u^\delta, v^\delta)$ such that

$$-\sigma_{ij,j} = f_i, \quad i = 1, 2, \quad \text{in} \ \ \Omega_\delta, \tag{1.8}$$

$$W^\delta = 0 \quad \text{on} \ \ \Gamma, \tag{1.9}$$

$$[W^\delta]n \geq 0 \quad \text{on} \ \ \Xi_{l+\delta}. \tag{1.10}$$

Here $\sigma_{ij} = \sigma_{ij}(W^\delta)$, $\varepsilon_{ij} = \varepsilon_{ij}(W^\delta)$, and σ_{ij}, ε_{ij} satisfy Hooke's law (1.4).

Similar to (1.1) – (1.3), the complete formulation of the problem (1.8) – (1.10) is variational ie., we minimise the functional

$$I(\Omega_\delta; U) = \frac{1}{2}\int\limits_{\Omega_\delta} \sigma_{ij}(U)\varepsilon_{ij}(U) - \int\limits_{\Omega_\delta} fU \tag{1.11}$$

over the set

$$K_\delta = \{(u, v) \in H^1(\Omega_\delta) \mid \quad u = v = 0 \quad \text{on} \ \ \Gamma, \quad [v] \geq 0 \quad \text{on} \ \ \Xi_{l+\delta}\}. \tag{1.12}$$

The inequality (1.10) is, in fact, only a part of the complete system of boundary conditions prescribed on $\Xi_{l+\delta}$.

We are going to establish the existence of the derivative of the energy functional

$$\lim_{\delta\to 0} \frac{I(\Omega_\delta; W^\delta) - I(\Omega; W)}{\delta}, \tag{1.13}$$

where W^δ, W are the solutions of (1.8) – (1.10) and (1.1) – (1.3), respectively. In this case the limit (1.13) is equal to the derivative

$$\frac{dJ(\Omega_\delta)}{d\delta}\bigg|_{\delta=0}, \tag{1.14}$$

where $J(\Omega_\delta) = I(\Omega_\delta; W^\delta)$.

To find the derivative (1.14), we transform the domain Ω_δ onto Ω in the following way. Consider a function $\theta \in C_0^\infty(D)$ such that $\theta = 1$ in a neighbourhood of the point $x_l = (l, 0)$. To simplify the arguments the function θ is assumed to be equal to zero in a neighbourhood of the point $(0, 0)$. Introduce the transformation of independent variables

$$y_1 = x_1 - \delta\theta(x_1, x_2), \quad y_2 = x_2. \tag{1.15}$$

Here $(y_1, y_2) \in \Omega$, $(x_1, x_2) \in \Omega_\delta$. The transformation (1.15) maps Ω_δ onto Ω, and it is one-to-one. The Jacobian q_δ of the transformation is positive for small δ,

$$q_\delta = \left|\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right| = 1 - \delta\theta_{x_1}.$$

Let $x = x(y, \delta)$ correspond to the transformation (1.15), and $W^\delta(x)$ be the solution of (1.8) – (1.10). Then $W^\delta(x) = W_\delta(y)$, $y \in \Omega$. Also, let W be the solution of (1.1) – (1.3). The following statement holds.

Lemma. *We have, as* $\delta \to 0$

$$\|W_\delta - W\|_{H^1(\Omega)} \ \to\ 0.$$

The proof is omitted here. The lemma can be proved by using the same arguments as those of Lemma 1 in (Khludnev, Sokolowski, 1998). We only indicate that W^δ and W are the solutions of the following variational inequalities

$$W^\delta \in K_\delta : \int\limits_{\Omega\delta} \sigma_{ij}(W^\delta)(\varepsilon_{ij}(V) - \varepsilon_{ij}(W^\delta)) \geq \int\limits_{\Omega_\delta} f(V - W^\delta) \quad \forall V \in K_\delta, \tag{1.16}$$

$$W \in K_0 : \int\limits_{\Omega} \sigma_{ij}(W)(\varepsilon_{ij}(V) - \varepsilon_{ij}(W)) \geq \int\limits_{\Omega} f(V - W) \quad \forall V \in K_0. \tag{1.17}$$

Moreover, similar to (Khludnev, Sokolowski, 1998) we can prove that there exists a constant $c > 0$ such that

$$\|W_\delta - W\|_{H^1(\Omega)} \leq c\delta.$$

Using the transformation (1.15) we obtain

$$\int\limits_{\Omega_\delta} f_i w^\delta \, dx = \int\limits_{\Omega} f_i^\delta(y)_\delta(y) \, dy, \quad f_i^\delta(y) = \frac{f_i(x(y,\delta))}{1 - \delta\theta_{x_1}}, \quad i = 1, 2, \;\; w^\delta(x) = w_\delta(y).$$

It is possible to find the derivatives

$$f_i'(y) = \lim_{\delta \to 0} \frac{f_i^\delta(y) - f_i^0(y)}{\delta} = \frac{df_i^\delta}{d\delta}\Big|_{\delta=0}.$$

In fact, assuming that y, δ are independent variables in (1.15), we have $x = x(y, \delta)$. Differentiation of (1.15) with respect to δ yields

$$0 = \frac{dx_1}{d\delta} - \theta - \delta\theta_{x_1}\frac{dx_1}{d\delta},$$

whence

$$\frac{dx_1}{d\delta} = \frac{\theta}{1 - \delta\theta_{x_1}}, \quad \frac{dx_2}{d\delta} = 0.$$

Consequently,

$$\frac{\partial f_i(x(y,\delta))}{\partial\delta}|_{\delta=0} = f_{ix_1}\frac{dx_1}{d\delta}|_{\delta=0} + f_{ix_2}\frac{dx_2}{d\delta}|_{\delta=0} = f_{iy_1}\theta, \; i = 1, 2. \tag{1.18}$$

Now we are in a position to find the derivative $f_i'(y)$. Indeed, by (1.18), we have

$$f_i'(y) = \lim_{\delta\to 0}\left(\frac{f_i(x(y,\delta))}{1 - \delta\theta_{x_1}} - f_i(y)\right)\frac{1}{\delta} = \lim_{\delta\to 0}\frac{f_i(x(y,\delta)) - f_i(y)}{\delta} +$$

$$+\theta_{x_1} f_i(y)|_{\delta=0} = f_{iy_1}\theta + \theta_{y_1} f_i = \frac{\partial}{\partial y_1}(\theta f_i),$$

i.e.

$$f_i'(y) = (\theta f_i)_{y_1}(y), \ i = 1, 2. \tag{1.19}$$

By $f_i \in C^1(\bar{\Omega})$, we can see that as $\delta \to 0$

$$\frac{f_i^\delta(y) - f_i^0(y)}{\delta} \to f_i'(y) \quad \text{in} \quad L^\infty(\Omega), \ i = 1, 2. \tag{1.20}$$

Convergence (1.20) is used in the sequel to find the derivative of the energy functional.

Let $W^\delta = (u^\delta, v^\delta)$ be the solution of the problem (1.8) – (1.10). Denote $u^\delta(x) = \tilde{u}(y)$, $v^\delta(x) = \tilde{v}(y)$, $x \in \Omega_\delta$, $y \in \Omega$, $x = x(y, \delta)$. Here, we use the tilde and omit δ for convenience.

Using (1.15), the following formulae are derived

$$\left\{ \begin{array}{l} u^\delta_{x_1} = \tilde{u}_{y_1}(1 - \delta\theta_{x_1}) \\ u^\delta_{x_2} = \tilde{u}_{y_1}(-\delta\theta_{x_2}) + \tilde{u}_{y_2}, \end{array} \right. \qquad \left\{ \begin{array}{l} v^\delta_{x_1} = \tilde{v}_{y_1}(1 - \delta\theta_{x_1}) \\ v^\delta_{x_2} = \tilde{v}_{y_1}(-\delta\theta_{x_2}) + \tilde{v}_{y_2}. \end{array} \right. \tag{1.21}$$

Since

$$\int\limits_{\Omega_\delta} \sigma_{ij}(W^\delta)\varepsilon_{ij}(W^\delta) = \int\limits_{\Omega_\delta} \Big((2\mu + \lambda)(\varepsilon_{11}^2(W^\delta) +$$

$$+\varepsilon_{22}^2(W^\delta)) + 2\lambda\varepsilon_{11}(W^\delta)\varepsilon_{22}(W^\delta) + 4\mu\varepsilon_{12}^2(W^\delta)\Big),$$

in view of (1.21), we can perform the change of variables and replace the domain of integration Ω_δ by Ω. This leads to

$$\frac{1}{2}\int\limits_{\Omega_\delta} \sigma_{ij}(W^\delta)\varepsilon_{ij}(W^\delta)dx - \int\limits_{\Omega_\delta} fW^\delta dx = \frac{1}{2}\int\limits_{\Omega} \frac{1}{q_\delta}\Big(\tilde{u}_{y_1}^2((2\mu+\lambda)(1-\delta\theta_{x_1})^2 - \mu\delta^2\theta_{x_2}^2) +$$

$$+\mu\tilde{u}_{y_2}^2 + \tilde{v}_{y_1}^2((2\mu + \lambda)\delta^2\theta_{x_2}^2 + \mu(1 - \delta\theta_{x_1})^2) + (2\mu + \lambda)\tilde{v}_{y_2}^2 + 2\mu\tilde{u}_{y_1}\tilde{u}_{y_2}(-\delta\theta_{x_2}) +$$

$$+2(\lambda + \mu)\tilde{u}_{y_1}\tilde{v}_{y_1}(-\delta\theta_{x_2})(1 - \delta\theta_{x_1}) + 2\lambda\tilde{u}_{y_1}\tilde{v}_{y_2}(1 - \delta\theta_{x_1}) + \tag{1.22}$$

$$+2\mu\tilde{u}_{y_2}\tilde{v}_{y_1}(1 - \delta\theta_{x_1}) + 2(2\mu + \lambda)\tilde{v}_{y_1}\tilde{v}_{y_2}(-\delta\theta_{x_2})\Big)dy - \int\limits_{\Omega} f^\delta \tilde{W} dy.$$

Denote by $I_\delta(\Omega;\tilde W)$ the right-hand side of (1.22), $\tilde W = (\tilde u, \tilde v)$. In this case, formula (1.22), actually, provides the transformation of the energy functional

$$I(\Omega_\delta; W^\delta) = I_\delta(\Omega; W_\delta), \tag{1.23}$$

where $W_\delta = \tilde W$. Again, let $W^\delta(x) = W_\delta(y)$, $x = x(y,\delta)$. Then $W_\delta \in K_0$ provided that $W^\delta \in K_\delta$ and conversely, the inclusion $W_\delta \in K_0$ implies $W^\delta \in K_\delta$. Thus we obtain a one-to-one mapping between K_δ and K_0. In particular, this implies

$$\min_{U\in K_\delta} I(\Omega_\delta; U) = \min_{U \in K_0} I_\delta(\Omega; U). \tag{1.24}$$

By (1.23), (1.24), we have

$$\frac{J(\Omega_\delta) - J(\Omega)}{\delta} = \frac{I(\Omega_\delta; W^\delta) - I(\Omega; W)}{\delta} = \frac{I_\delta(\Omega; W_\delta) - I(\Omega; W)}{\delta} \le$$

$$\le \frac{I_\delta(\Omega; W) - I(\Omega; W)}{\delta},$$

whence

$$\limsup_{\delta\to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \le \limsup_{\delta\to 0} \frac{I_\delta(\Omega; W) - I(\Omega; W)}{\delta}. \tag{1.25}$$

On the other hand

$$\liminf_{\delta\to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \ge \liminf_{\delta\to 0} \frac{I_\delta(\Omega; W_\delta) - I(\Omega; W_\delta)}{\delta}. \tag{1.26}$$

Taking into account (1.22), (1.20) and Lemma, we can show that the right-hand sides of (1.25) and (1.26) coincide. Consequently, there exists a limit

$$\lim_{\delta\to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta}$$

which proves the existence of the derivative (1.14). Direct calculation of the right-hand sides of (1.25), (1.26) implies that

$$\left.\frac{dJ(\Omega_\delta)}{d\delta}\right|_{\delta=0} = \frac{1}{2}\int_\Omega \Big((2\mu+\lambda)u_{y_1}^2(-\theta_{y_1}) + \mu u_{y_2}^2\theta_{y_1} + 2\mu u_{y_1}u_{y_2}(-\theta_{y_2}) +$$

$$+\mu v_{y_1}^2(-\theta_{y_1}) + (2\mu+\lambda)v_{y_2}^2\theta_{y_1} + 2(\lambda+\mu)u_{y_1}v_{y_1}(-\theta_{y_2}) + \tag{1.27}$$
$$+2(2\mu+\lambda)v_{y_1}v_{y_2}(-\theta_{y_2})\Big) - \int_\Omega (\theta f_1)_{y_1} u - \int_\Omega (\theta f_2)_{y_1} v.$$

As a result, the Griffith formula (1.27) is derived. The formula represents the derivative of the energy functional with respect to the crack length for two-dimensional elasticity with nonlinear boundary conditions (1.28) below. Derivatives of the energy functional for the Poisson equation and the linear elasticity equations with linear boundary conditions prescribed on Ξ_l have been studied in (Mazja and Nazarov, 1987), (Grisvard, 1992), (see also (Bui and Ehrlacher, 1997), (Ohtsuka, 1994), (Blat and Morel, 1988), (Destuynder and Jaoua, 1981), (Bonnet, 1994)). Regularity of solutions to elliptic equations in nonsmooth domains are analysed in (Khludnev, 1995, 1996a, 1996b), (Kondratiev *et al.*, 1982). Other aspects of elliptic problems in domains with nonsmooth boundaries can be found in (Khludnev and Sokolowski, 1997), (Grisvard, 1992), (Dauge, 1988), (Nazarov and Plamenevskii, 1991). First, we analyse boundary conditions prescribed on Ξ_l for the problem (1.1) – (1.3). According to (Khludnev and Sokolowski, 1997), the following conditions hold at Ξ_l

$$[v] \geq 0, \quad \sigma_{22} \leq 0, \quad [\sigma_{22}] = 0, \quad \sigma_{12} = 0, \quad [v]\sigma_{22} = 0. \tag{1.28}$$

Moreover, by (Khludnev, 1996a), the solution of the problem (1.1) – (1.3) (i.e., in fact, the problem (1.17)) has additional regularity properties up to the crack faces. Namely, for any $x \in \Xi_l$ there exists a neighbourhood V of the point x such that $W \in H^2(V \backslash \Xi_l)$. Consequently, the solution W is continuous up to the crack faces, and the conditions (1.28) hold almost everywhere at Ξ_l. Note that

$$\sigma_{22} = (2\mu+\lambda)v_{y_2} + \lambda u_{y_1}, \quad \sigma_{12} = \mu(u_{y_2} + v_{y_1}). \tag{1.29}$$

In addition to (1.28) we can prove that

$$[\sigma_{22} v_{y_1}] = \sigma_{22}[v_{y_1}] = 0 \quad \text{a.e. on } \Xi_l. \tag{1.30}$$

Indeed, since v is continuous the set

$$M = \{y \in \Xi_l \mid \quad [v(y)] > 0\}$$

is open. At any point $y \in M$ we have $[v(y)] > 0$, therefore, by the last equality in (1.28), $\sigma_{22}(y) = 0$, and consequently $\sigma_{22}[v_{y_1}] = 0$ a.e. on M. We have $[v] = 0$ on the set $\Xi_l \backslash M$, whence $[v_{y_1}] = 0$ (see Kinderlehrer and Stampacchia, 1980, Chapter 2, Theorem A.1) which implies $\sigma_{22}[v_{y_1}] = 0$. The equality (1.30) is proved.

Now we prove that the right-hand side of (1.27) is independent of θ. Consider two right-hand sides corresponding to two functions θ_1, θ_2 with the required properties. Denote by Λ the difference between the right-hand sides of (1.27). We have

$$\Lambda = \frac{1}{2} \int\limits_{\Omega} \Big((2\mu+\lambda) u_{y_1}^2 (-\theta_{y_1}) + \mu u_{y_2}^2 \theta_{y_1} + 2\mu u_{y_1} u_{y_2} (-\theta_{y_2}) + \\ + \mu v_{y_1}^2 (-\theta_{y_1}) + (2\mu+\lambda) v_{y_2}^2 \theta_{y_1} + 2(\lambda+\mu) u_{y_1} v_{y_1} (-\theta_{y_2}) + \\ + 2(2\mu+\lambda) v_{y_1} v_{y_2} (-\theta_{y_2}) \Big) - \int\limits_{\Omega} (\theta f_1)_{y_1} u - \int\limits_{\Omega} (\theta f_2)_{y_1}) v, \tag{1.31}$$

where $\theta = \theta_1 - \theta_2$. The functions θ_1, θ_2 are equal to 1 in neighbourhoods of the point x_l, consequently, the integration in (1.31) is actually fulfilled over $\Omega \backslash B_{x_l}$, where B_{x_l} is a ball of small radius centred at x_l. Integrating by parts in (1.31) leads to

$$\Lambda = \int\limits_{\Omega \backslash B_{x_l}} \theta \Big((2\mu+\lambda) u_{y_1} u_{y_1 y_1} + \mu u_{y_1} u_{y_2 y_2} + \mu v_{y_1} v_{y_1 y_1} + (\lambda+\mu) v_{y_1} u_{y_1 y_2} + \\ + (\lambda+\mu) u_{y_1} v_{y_1 y_2} + (2\mu+\lambda) v_{y_1} v_{y_2 y_2} + f_1 u_{y_1} + f_2 v_{y_1} \Big) + \\ + \int\limits_{\Xi_l} \theta \left(\mu [u_{y_1} u_{y_2}] + (\lambda+\mu) [u_{y_1} v_{y_1}] + (2\mu+\lambda) [v_{y_1} v_{y_2}] \right). \tag{1.32}$$

Note that the equilibrium equations (1.1) can be written as

$$\begin{aligned} (2\mu+\lambda) u_{y_1 y_1} + \mu u_{y_2 y_2} + (\lambda+\mu) v_{y_1 y_2} &= -f_1, \\ (2\mu+\lambda) v_{y_2 y_2} + (\lambda+\mu) u_{y_1 y_2} + \mu v_{y_1 y_1} &= -f_2, \end{aligned} \tag{1.33}$$

and

$$\mu [u_{y_1} u_{y_2}] + (\lambda+\mu) [u_{y_1} v_{y_1}] + (2\mu+\lambda) [v_{y_1} v_{y_2}] = [\sigma_{12} u_{y_1}] + [\sigma_{22} v_{y_1}]. \tag{1.34}$$

By (1.28) – (1.30), (1.33), the right-hand side of (1.32) is equal to zero which proves the independence of the right-hand side of (1.27) on θ. Note that the independence, actually, follows by simpler arguments. In fact, we have

$$\liminf_{\delta\to 0} \frac{J(\Omega_\delta)-J(\Omega)}{\delta} = \limsup_{\delta\to 0} \frac{J(\Omega_\delta)-J(\Omega)}{\delta} \tag{1.35}$$

which proves the existence $dJ(\Omega_\delta)/d\delta|_{\delta=0}$. Both sides of (1.35) are independent of θ, hence $\lim_{\delta\to 0}(J(\Omega_\delta)-J(\Omega))\delta^{-1}$ exists and is independent of θ.

In some particular cases we can prove an additional regularity of the solution. For instance, assume that the solution W of the problem (1.1) – (1.3) has the property

$$[W]=0 \quad \text{on} \quad B_{x_l}\cap\Xi_l,$$

where B_{x_l} is any ball centred at x_l. Then the arguments used in (Khludnev, 1995) to prove C^∞- regularity of the solution can be used to show that the equations

$$-\sigma_{ij,j}(W)=f_i, \quad i=1,2,$$

are satisfied in B_{x_l} in the sense of distributions. Consequently, $W\in H^3_{loc}(B_{x_l})$. In addition, by the inclusion $f\in H^1(D)$, we have $W\in H^3_{loc}(\Omega)$. In this case

$$\frac{dJ(\Omega_l)}{dl}=0, \tag{1.36}$$

where we denote the domain Ω by Ω_l. In fact, we can integrate by parts in the right-hand side of (1.27) which gives

$$\frac{dJ(\Omega_l)}{dl} = \int_\Omega \theta\left((\sigma_{11,1}+\sigma_{12,2})u_{y_1}+(\sigma_{21,1}+\sigma_{22,2})v_{y_1}+f_1u_{y_1}+f_2v_{y_1}\right)+ \tag{1.37}$$

$$+\int_{\Xi_l}\theta\left(\sigma_{22}[v_{y_1}]+[\sigma_{12}u_{y_1}]\right).$$

By (1.1), (1.28), (1.30), the right-hand side of (1.37) is equal to zero and (1.36) follows.

The regularity $W\in H^2(B_{x_l}\backslash\Xi_l)$ is in fact sufficient to justify (1.36). In this case we can repeat the above arguments and obtain that the right-hand side of (1.37) is equal to zero.

The formula (1.27) can be written in the form which does not contain the function θ. To show this we choose a ball $B_{x_l}(r)$ of radius r with the boundary $\Gamma(r)$ such that $\theta = 1$ on $B_{x_l}(r)$. In this case the integration by parts in (1.27) yields

$$\begin{aligned} \frac{dJ(\Omega_l)}{dl} = & \int\limits_{B_{x_l}(r)\setminus\Xi_l} (f_1 u_{y_1} + f_2 v_{y_1}) + \frac{1}{2}\int\limits_{\Gamma(r)} \nu_1 \Big((2\mu+\lambda)(u_{y_1}^2 - v_{y_2}^2) + \mu(v_{y_1}^2 - u_{y_2}^2)\Big) \\ & + \int\limits_{\Gamma(r)} \nu_2\Big((2\mu+\lambda)v_{y_1}v_{y_2} + (\lambda+\mu)u_{y_1}v_{y_1} + \mu u_{y_1}u_{y_2}\Big)\,, \end{aligned} \tag{1.38}$$

where (ν_1, ν_2) is the unit normal exterior vector to $\Gamma(r)$.

Now assume that $f = 0$ in some neighbourhood V of the point x_l. For sufficiently small r, $0 < r < r_0$, we have $B_{x_l}(r) \subset V$, and the formula (1.38) implies

$$\begin{aligned} \frac{dJ(\Omega_l)}{dl} = & \frac{1}{2}\int\limits_{\Gamma(r)} \nu_1 \Big((2\mu+\lambda)(u_{y_1}^2 - v_{y_2}^2) + \mu(v_{y_1}^2 - u_{y_2}^2)\Big) + \\ & + \int\limits_{\Gamma(r)} \nu_2\Big((2\mu+\lambda)v_{y_1}v_{y_2} + (\lambda+\mu)u_{y_1}v_{y_1} + \mu u_{y_1}u_{y_2}\Big)\,. \end{aligned} \tag{1.39}$$

The right-hand side of (1.39) does not depend on r, consequently, we have the following property. Let W be the solution of the problem (1.1)– (1.3), and f be equal to zero in some neighbourhood of the point x_l. Then the integral

$$\begin{aligned} I = & \frac{1}{2}\int\limits_{\Gamma(r)} \nu_1 \Big((2\mu+\lambda)(u_{y_1}^2 - v_{y_2}^2) + \mu(v_{y_1}^2 - u_{y_2}^2)\Big) + \\ & + \int\limits_{\Gamma(r)} \nu_2\Big((2\mu+\lambda)v_{y_1}v_{y_2} + (\lambda+\mu)u_{y_1}v_{y_1} + \mu u_{y_1}u_{y_2}\Big) \end{aligned}$$

is independent of $r > 0$ for all sufficiently small r. Moreover, the above arguments show that the integral

$$I = \frac{1}{2}\int\limits_C \nu_1 \Big((2\mu+\lambda)(u^2_{y_1} - v^2_{y_2}) + \mu(v^2_{y_1} - u^2_{y_2})\Big) + \tag{1.40}$$

$$+\int\limits_C \nu_2\Big((2\mu+\lambda)v_{y_1}v_{y_2} + (\lambda+\mu)u_{y_1}v_{y_1} + \mu u_{y_1}u_{y_2}\Big)$$

is path independent for any closed curve C surrounding the point x_l. In this case $\nu = (\nu_1, \nu_2)$ is the normal unit vector to the curve C. Some part of this curve denoted by $\Xi = \Xi_l \cap C$ may belong to Ξ_l. In this case, taking into account (1.34), (1.30), (1.28), we can integrate over Ξ^+ or Ξ^- in (1.40) and the same value of the integral is obtained. Of course, the above independence takes place provided that f is equal to zero in a domain with the boundary C.

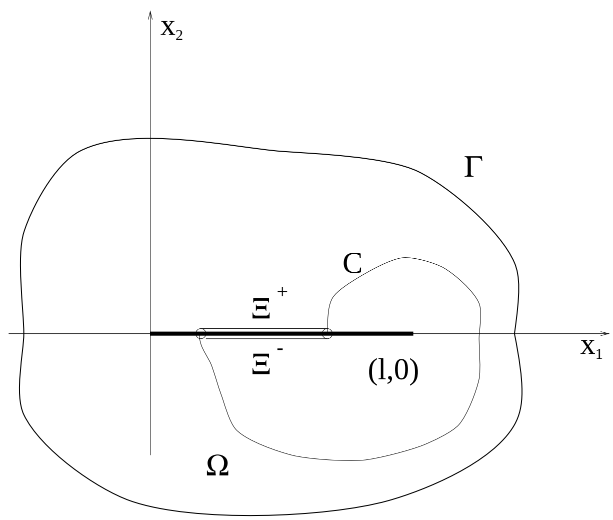

Figure 2: Curve C.

The integral of the form (1.40) is called the Rice–Cherepanov's integral. We have to note that the result is obtained for nonlinear boundary conditions (1.28). The well-known path independence of the Rice–Cherepanov's integral was previously proved in elasticity theory for linear boundary conditions $\sigma_{22} = 0, \sigma_{12} = 0$ holding on $\Xi^\pm_l$ (see (Parton and Morozov, 1985)).

2. Three-dimensional case.

Let $D \subset R^3$ be a bounded domain with a smooth boundary Γ. The crack is defined in the form of the two-dimensional surface,

$$\Xi_{l+\delta} = \{(x_1, x_2, x_3) \mid \quad x_3 = 0, \quad -h < x_2 < h, \quad 0 < x_1 < l + \delta\}.$$

Here, $h > 0$, $l > 0$, δ is a small parameter which converges to zero. Assume that $\Xi_{l+\delta} \subset D$ and denote $\Omega_\delta = D \backslash \overline{\Xi}_{l+\delta}$, $\Omega = D \backslash \overline{\Xi}_l$.

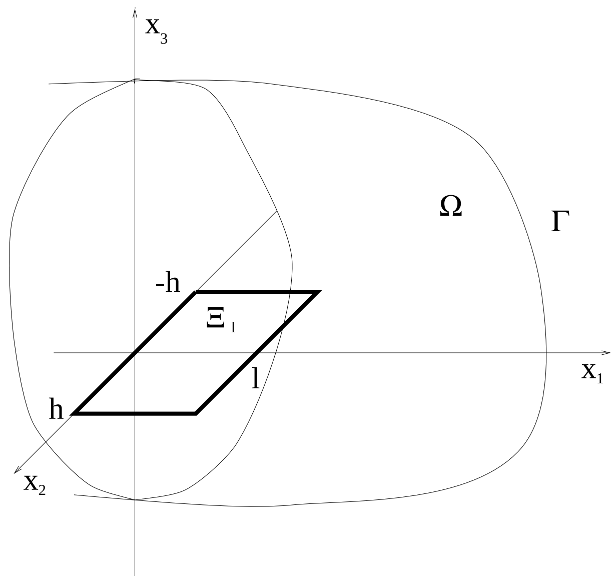

Figure 3: Domain Ω and crack Ξ_l in R^3.

The equilibrium problem for an elastic body occupying the domain Ω_δ can be formulated as follows. We want to find a function $W = (u, v, w)$ such that

$$-\sigma_{ij,j} = f_i, \quad i = 1, 2, 3, \quad \text{in} \ \ \Omega, \tag{2.1}$$

$$W = 0 \quad \text{on} \ \ \Gamma, \tag{2.2}$$

$$[W]n \geq 0 \quad \text{on} \ \ \Xi_l. \tag{2.3}$$

Here $\sigma_{ij} = \sigma_{ij}(W)$ are stress tensor components, $n = (0, 0, 1)$ is a normal vector to the surface $\Xi_{l+\delta}$. We assume that Hooke's law is fulfilled,

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda \operatorname{div} W \, \delta^i_j, \quad i, j = 1, 2, 3, \tag{2.4}$$

$$\varepsilon_{ij}(W) = \frac{1}{2}(w^i_{,j} + w^j_{,i}), \quad (w^1, w^2, w^3) \equiv (u, v, w),$$

where λ, μ are the Lame parameters, δ^i_j is the Kronecker symbol.

Actually, by considering the problem (2.1) – (2.3) we have in mind the minimisation of the functional

$$I(\Omega; U) = \frac{1}{2}\int\limits_{\Omega} \sigma_{ij}(U)\varepsilon_{ij}(U) - \int\limits_{\Omega} fU \ , \quad U = (u, v, w), \tag{2.5}$$

over the set

$$K_0 = \{(u, v, w) \in H^1(\Omega) \mid u = v = w = 0 \ \text{ on } \ \Gamma, \ [w] \geq 0 \ \text{ on } \ \Xi_l\}, \tag{2.6}$$

where $f = (f_1, f_2, f_3) \in C^1(\bar{D})$. In this case the conditions (2.1) – (2.3) are satisfied. Moreover, a system of equations and inequalities holds on Ξ_l, and (2.3) is a part of this system (see (2.22) below).

The perturbed problem corresponding to (2.1) – (2.3) is defined as follows. We want to find a function $W^\delta = (u^\delta, v^\delta, w^\delta)$ such that

$$-\sigma_{ij,j} = f_i, \quad i = 1, 2, 3, \quad \text{ in } \ \Omega_\delta, \tag{2.7}$$

$$W^\delta = 0 \quad \text{ on } \ \Gamma, \tag{2.8}$$

$$[W^\delta]n \geq 0 \quad \text{ on } \ \Xi_{l+\delta}. \tag{2.9}$$

Here $\sigma_{ij} = \sigma_{ij}(W^\delta)$, and $\sigma_{ij}(W^\delta)$, $\varepsilon_{ij}(W^\delta)$ satisfy Hooke's law (2.4). Analogously, by considering the problem (2.7) – (2.9) we, in fact, consider the minimisation of the energy functional

$$I(\Omega_\delta; U) = \frac{1}{2}\int\limits_{\Omega_\delta} \sigma_{ij}(U)\varepsilon_{ij}(U) - \int\limits_{\Omega_\delta} fU \tag{2.10}$$

over the set

$$K_\delta = \{(u, v, w) \in H^1(\Omega_\delta) \mid u = v = w = 0 \ \text{ on } \ \Gamma, \ [w] \geq 0 \ \text{ on } \ \Xi_{l+\delta}\}. \tag{2.11}$$

Denote by $J(\Omega_\delta) = I(\Omega_\delta; W^\delta)$, $J(\Omega) = I(\Omega; W)$ the energy functionals, where W^δ, W are the solutions of (2.1) – (2.3) and (2.7) – (2.9), respectively. In this section, we determine the derivative

$$\lim_{\delta \to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta}. \tag{2.12}$$

Consider a function $\theta \in C_0^\infty(D)$, $\theta = 1$ in a neighbourhood of the set L,

$$L = \{(x_1, x_2, x_3)|\ x_1 = l,\ -h < x_2 < h,\ x_3 = 0\}.$$

The transformation of the variables

$$y_1 = x_1 - \delta\theta(x_1, x_2, x_3), \quad y_2 = x_2, \quad y_3 = x_3 \tag{2.13}$$

maps Ω_δ on Ω. The Jacobian q_δ is positive for all sufficiently small δ,

$$q_\delta = \left|\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}\right| = 1 - \delta\theta_{x_1}, \tag{2.14}$$

hence the transformation is one-to-one. For simplicity, the function θ is assumed to be equal to zero in a neighbourhood of the set $\{(x_1, x_2, x_3)|\ x_1 = 0,\ -h < x_2 < h,\ x_3 = 0\}$. Let $(u^\delta, v^\delta, w^\delta)$ be the solution of the problem (2.7) – (2.9). Denote $(u^\delta(x), v^\delta(x), w^\delta(x)) = (\tilde{u}(y), \tilde{v}(y), \tilde{w}(y))$. Let $x = x(y, \delta)$ be the inverse transformation for the transformation (2.13). We use the tilde instead of δ to simplify the formulae below. By (2.13), the following relations hold

$$u^\delta_{x_1} = \tilde{u}_{y_1}(1 - \delta\theta_{x_1}),\ \ u^\delta_{x_2} = \tilde{u}_{y_1}(-\delta\theta_{x_2}) + \tilde{u}_{y_2},\ \ u^\delta_{x_3} = \tilde{u}_{y_1}(-\delta\theta_{x_3}) + \tilde{u}_{y_3}. \tag{2.15}$$

The similar formulae are valid for the functions $v^\delta(x)$, $w^\delta(x)$.

Let $\sigma_{ij} = \sigma_{ij}(W^\delta)$, $\varepsilon_{ij} = \varepsilon_{ij}(W^\delta)$. In this case, by (2.4),

$$\sigma_{ij}\varepsilon_{ij} = (2\mu+\lambda)(\varepsilon_{11}^2+\varepsilon_{22}^2+\varepsilon_{33}^2)+4\mu(\varepsilon_{12}^2+\varepsilon_{13}^2+\varepsilon_{23}^2)+2\lambda(\varepsilon_{11}\varepsilon_{22}+\varepsilon_{22}\varepsilon_{33}+\varepsilon_{11}\varepsilon_{33}).$$

This allows us to obtain the transformation of the energy functional,

$$\frac{1}{2}\int\limits_{\Omega_\delta} \sigma_{ij}\varepsilon_{ij} - \int\limits_{\Omega_\delta} fW^\delta = \frac{1}{2}\int\limits_{\Omega} \frac{1}{q_\delta}\Big((2\mu + \lambda)((1 - \delta\theta_{x_1})^2\tilde{u}_{y_1}^2 + (\tilde{v}_{y_1}(-\delta\theta_{x_2}) + \tilde{v}_{y_2})^2+$$

$$+(\tilde{w}_{y_1}(-\delta\theta_{x_3}) + \tilde{w}_{y_3})^2) + \mu((\tilde{u}_{y_1}(-\delta\theta_{x_2}) + \tilde{u}_{y_2} + \tilde{v}_{y_1}(1 - \delta\theta_{x_1}))^2 + (\tilde{u}_{y_1}(-\delta\theta_{x_3})+$$

$$+\tilde{u}_{y_3} + \tilde{w}_{y_1}(1 - \delta\theta_{x_1}))^2 + (\tilde{v}_{y_1}(-\delta\theta_{x_3}) + \tilde{v}_{y_3} + \tilde{w}_{y_1}(-\delta\theta_{x_2}) + \tilde{w}_{y_2})^2)+$$

$$+2\lambda(\tilde{u}_{y_1}(1 - \delta\theta_{x_1})(\tilde{v}_{y_1}(-\delta\theta_{x_2}) + \tilde{v}_{y_2}) + (\tilde{v}_{y_1}(-\delta\theta_{x_2}) + \tilde{v}_{y_2})(\tilde{w}_{y_1}(-\delta\theta_{x_3})+$$

$$+\tilde{w}_{y_3}) + \tilde{u}_{y_1}(1-\delta\theta_{x_1})(\tilde{w}_{y_1}(-\delta\theta_{x_3}) + \tilde{w}_{y_3}))\Big) - \int_\Omega f^\delta \tilde{W}, \tag{2.16}$$

where

$$f^\delta(y) = \frac{f(x(y,\delta))}{1-\delta\theta_{x_1}}, \quad \tilde{W} = (\tilde{u},\tilde{v},\tilde{w}) = W_\delta.$$

Consequently, we have the following equality

$$I(\Omega_\delta; W^\delta) = I_\delta(\Omega; W_\delta). \tag{2.17}$$

Now we can proceed in the same way as in the previous section, which gives

$$\limsup_{\delta\to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \le \limsup_{\delta\to 0} \frac{I_\delta(\Omega; W) - I(\Omega; W)}{\delta}, \tag{2.18}$$

$$\liminf_{\delta\to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \ge \liminf_{\delta\to 0} \frac{I_\delta(\Omega; W_\delta) - I(\Omega; W_\delta)}{\delta}. \tag{2.19}$$

Since the right-hand sides of (2.18), (2.19) coincide, we obtain the existence of limit (2.12) in the form of the following Griffith formula

$$\frac{dJ(\Omega_\delta)}{d\delta}\Big|_{\delta=0} = \frac{1}{2}\int_\Omega \Big((2\mu+\lambda)(u^2_{y_1}(-\theta_{y_1}) + v^2_{y_2}\theta_{y_1} + w^2_{y_3}\theta_{y_1} + 2v_{y_1}v_{y_2}(-\theta_{y_2}) +$$

$$+2w_{y_1}w_{y_3}(-\theta_{y_3})) + \mu(u^2_{y_2}\theta_{y_1} + v^2_{y_1}(-\theta_{y_1}) + u^2_{y_3}\theta_{y_1} + w^2_{y_1}(-\theta_{y_1}) + v^2_{y_3}\theta_{y_1} +$$

$$+w^2_{y_2}\theta_{y_1} + 2u_{y_1}u_{y_3}(-\theta_{y_3}) + 2u_{y_1}w_{y_1}(-\theta_{y_3}) + 2v_{y_1}v_{y_3}(-\theta_{y_3}) + 2v_{y_1}w_{y_2}(-\theta_{y_3}) +$$

$$+2v_{y_3}w_{y_1}(-\theta_{y_2}) + 2w_{y_1}w_{y_2}(-\theta_{y_2}) + 2u_{y_1}u_{y_2}(-\theta_{y_2}) + 2v_{y_1}u_{y_1}(-\theta_{y_2}) + 2v_{y_3}w_{y_2}\theta_{y_1}) +$$

$$+2\lambda(v_{y_2}w_{y_3}\theta_{y_1} + u_{y_1}w_{y_1}(-\theta_{y_3}) + v_{y_2}w_{y_1}(-\theta_{y_3}) + v_{y_1}w_{y_3}(-\theta_{y_2}) + u_{y_1}v_{y_1}(-\theta_{y_2}))\Big) -$$

$$-\int_\Omega (\theta f_1)_{y_1} u - \int_\Omega (\theta f_2)_{y_1} v - \int_\Omega (\theta f_3)_{y_1} w. \tag{2.20}$$

It is not difficult to show that the right-hand side of (2.20) does not depend on θ. To prove this, consider the difference between right-hand sides of (2.20) corresponding to any two functions θ_1, θ_2. Let $\theta = \theta_1 - \theta_2$. We integrate by parts in (2.20) which implies that the difference denoted by Λ between the right-hand sides of (2.20) evaluated for θ_1,θ_2, respectively, is equal to

$$\Lambda = \int\limits_{\Omega\setminus S_L} \theta\left((\sigma_{1j,j} + f_1)u_{,1} + (\sigma_{2j,j} + f_2)v_{,1} + (\sigma_{3j,j} + f_3)w_{,1}\right) +$$

$$+ \int\limits_{\Xi_l\setminus S_L} \theta\Big((2\mu + \lambda)[w_{,1}w_{,3}] + \mu[u_{,1}u_{,3}] + \mu[u_{,1}w_{,1}]+ \tag{2.21}$$

$$+\mu[v_{,1}v_{,3}] + \mu[v_{,1}w_{,2}] + \lambda[u_{,1}w_{,1}] + \lambda[v_{,2}w_{,1}]\Big).$$

We should remind at this point that $\theta = \theta_1 - \theta_2 = 0$ in some neighbourhood of L. This neighbourhood is denoted by S_L. It is known that the solution of the problem (2.1) – (2.3) has an additional regularity up to the crack faces (see (Khludnev and Sokolowski, 1997)). For any point $x \in \Xi_l$ there exists a neighbourhood V of the point x such that

$$W \in H^2(V \setminus \Xi_l).$$

In particular, by the Sobolev imbedding theorem, W is continuous up to the crack faces. As it was shown in (Khludnev and Sokolowski, 1997), the solution W satisfies the following boundary condition on Ξ_l

$$[w] \geq 0, \quad [\sigma_{33}] = 0, \quad \sigma_{33} \leq 0, \quad [w]\sigma_{33} = 0, \quad \sigma_{13} = 0, \quad \sigma_{23} = 0. \tag{2.22}$$

It is easy to see that

$$\sigma_{33} = 2\mu w_{,3}+\lambda(u_{,1}+v_{,2}+w_{,3}), \quad \sigma_{13} = \mu(u_{,3}+w_{,1}), \quad \sigma_{23} = \mu(v_{,3}+w_{,2}). \tag{2.23}$$

From (2.21), in view of (2.1), (2.23), it follows that

$$\Lambda = \int\limits_{\Xi_l\setminus S_L} \theta\left([\sigma_{33}w_{,1}] + [\sigma_{13}u_{,1}] + [\sigma_{23}v_{,1}]\right) \tag{2.24}$$

and, consequently, by (2.22),

$$\Lambda = \int\limits_{\Xi_l\setminus S_L} \theta\sigma_{33}[w_{,1}].$$

Let us prove that

$$\sigma_{33}[w_{,1}] = 0 \quad \text{a.e. on} \quad \Xi_l \setminus S_L. \tag{2.25}$$

Using (2.22) we have $\sigma_{33}(y) = 0$ on the set

$$M = \{y \in \Xi_l \setminus S_L \;\Big|\; [w(y)] > 0\}$$

and, consequently, $\sigma_{33}(y)[w_{,1}(y)] = 0$. The complement $(\Xi_l \setminus S_L) \setminus M$ is characterised by the condition

$$[w(y)] = 0,$$

hence $[w_{,1}(y)] = 0$ a.e. on $(\Xi_l \setminus S_L) \setminus M$ (see Kinderlehrer and Stampacchia, 1980 , Chapter 2, Theorem A.1). Thus $\sigma_{33}[w_{,1}] = 0$ a.e. on $(\Xi_l \setminus S_L) \setminus M$. As a consequence, we obtain (2.25), and hence $\Lambda = 0$ which shows the independence of the right-hand side in (2.20) on θ.

To conclude the section we write the formula (2.20) in the form which does not contain the function θ. To this end, consider a neighbourhood S_L of the set L with a smooth boundary Γ_L and assume that $\theta = 1$ on S_L. Denote by (ν_1, ν_2, ν_3) the unit external normal vector to Γ_L. Integrating by parts in (2.20) we obtain

$$\begin{aligned}
\frac{dJ(\Omega_\delta)}{d\delta}\Big|_{\delta=0} = \int_{S_L} (f_1 u_{y_1} + f_2 v_{y_1} + f_3 w_{y_1}) + \frac{1}{2}\int_{\Gamma_L} \nu_1\Big((2\mu+\lambda)(u_{y_1}^2 - v_{y_2}^2 - w_{y_3}^2) + \\
+\mu(v_{y_1}^2 - u_{y_2}^2 - u_{y_3}^2 + w_{y_1}^2 - v_{y_3}^2 - w_{y_2}^2 - 2v_{y_3}w_{y_2}) - 2\lambda v_{y_2}w_{y_3}\Big) + \\
+\int_{\Gamma_L} \nu_2\Big((2\mu+\lambda)v_{y_1}v_{y_2} + \mu(v_{y_3}w_{y_1} + w_{y_1}w_{y_2} + u_{y_1}u_{y_2} + v_{y_1}u_{y_1}) + \\
+\lambda(v_{y_1}w_{y_3} + u_{y_1}v_{y_1})\Big) + \int_{\Gamma_L} \nu_3\Big((2\mu+\lambda)w_{y_1}w_{y_3} + \\
+\mu(u_{y_1}u_{y_3} + u_{y_1}w_{y_1} + v_{y_1}v_{y_3} + v_{y_1}w_{y_2}) + \lambda(u_{y_1}w_{y_1} + v_{y_2}w_{y_1})\Big).
\end{aligned} \tag{2.26}$$

Denoting by $k(l,h,f)$ the functional defined by the right-hand side of (2.26) we have

$$\frac{dJ(\Omega_\delta)}{d\delta}\Big|_{\delta=0} = k(l,h,f).$$

Therefore,

$$J(\Omega_\delta) = J(\Omega) + k(l,h,f)\delta + \alpha(\delta)\delta,$$

where $\alpha(\delta) \to 0$ as $\delta \to 0$. Note that $k(l,h,f)$ is independent of S_L.

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